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# Eikonal type equations for geometrical singularities of solutions in field theory

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**Abstract**

We discuss several aspects of singularities of the solutions of the partial differential equations of Klein–Gordon, Schrödinger and Dirac. In particular we analyze the fold-type singularity, of the first and higher orders, and the related characteristic equations. We also consider the field equations as reduction of homogeneous equations in higher dimensions, and discuss how singularities of the solution are reduced.

*Key words:* Partial differential equations, (Geometric) solution singularities, Characteristic, eikonal and Hamilton–Jacobi equations, Field (Klein–Gordon, Schrödinger, Dirac, Maxwell) equations, $k$-characteristic equations, Complementary equations,
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## 1. Introduction

The Lagrangian formalism plays a major role in the description of evolutionary systems in Physics. Among other things, it allows for manifestly covariant theories, Noether's theorem and locality.

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Many relevant Lagrangians for physics (gauge theories, gravitation, relativistic particles) give rise to dynamical systems in implicit form, i.e. they do not give rise to *vector fields*. They only determine a submanifold of the relevant carrier space, and this submanifold need not be a section of the appropriate bundle. For these systems one usually deals with constrained formalism, as elaborated by Dirac and Bergman. However, this procedure does not appear to be a natural approach to these equations, for one is forced to deal with the inverse of a matrix which may change rank from point to point. This equations are instances of implicit differential equations and their solutions may exhibit singularities.

Another familiar example of partial differential equations (PDEs) arising in implicit form in physics is provided by the Hamilton–Jacobi form of dynamics. Here the equation for a function $S$ on the configuration space $Q$ has the form

$$H\left(q, \frac{\partial S}{\partial q}\right) - \frac{\partial S}{\partial t} = 0 \quad .$$

The Hamiltonian function $H = H(q,p,t)$ is in general non-linear and gives rise to an implicit differential equation for $S$. In more geometrical terms, the equation for $S$ is replaced by an equation for Lagrangian submanifolds which are not necessarily sections of the cotangent bundle $T^*(Q) \xrightarrow{\pi} Q$. To simply illustrate the situation we restrict to $Q = R^n$, $T^*(Q) = R^{2n}$. Given a Hamiltonian function $H$, the associated generalized version of the Hamilton–Jacobi form of the dynamics can be given along the following lines.

First find the embedding $\Phi$ :

$$\begin{array}{ccc} R^n & \xrightarrow{\Phi} & T^*R^n \\ & & \downarrow \\ & & R^n \end{array}$$

with the property

$$\Phi^* H = E, \qquad \Phi^* \omega_0 = 0$$

where $\omega_0 = dp_i \wedge dq^i$ is the canonical symplectic structure. Of course if $\Phi(R^n)$ is transverse to the fibers of $T^*R^n$, we can find locally a function $S$ such that $dS(R^n) = \Phi(R^n)$ in the appropriate neighborhood. In many cases solutions $\Phi$ will fail to be transverse to the fibers, caustics arise in this way. Other singularities also show up in this respect.

When the symplectic structure is replaced by the contact structure on $R^{2n+1}$, the 1-jets of functions on $R^n$, we have Legendre rather than Lagrange embeddings. The projection of this submanifold on the base may not be a diffeomorphism. The set where there is lack of transversality is the wave front. The connection between Hamilton–Jacobi theory and the Schrödinger equation shows that the analysis of these singularities is very important in the WKB approximation of quasi-classical asymptotics of the solutions of the Schrödinger equation. One can hypothesize that the geometric background found by V. Maslov [1] for quasi-classical asymptotic solutions gives rise to a similar theory at the level of exact solutions.

The study of these singularities is centered around the subsidiary equations, describing all possible forms of a prescribed type of singularities admitted by a given system of PDE's. Therefore it is necessary to develop a theory of singular solutions of PDE's. Two steps are needed:

(i) The first step is to formalize the concept of singularity for solutions of PDE's, and to classify them.

(ii) The second step requires that we develop a formal procedure to associate, with a given system of PDE's $\mathcal{Y}$ and a given singularity type $\Sigma$, the subsidiary equations ($\mathcal{Y}_\Sigma$) mentioned above.

Also central in this approach is the reconstruction problem, that is, given the system of equations $\mathcal{Y}_\Sigma$, and the singularity type $\Sigma$ (to which they correspond), find the original system $\mathcal{Y}$. The quantization procedure, as well as the problem of the sources of the fields, are of this kind. It also seems to be very important for the mechanics of continuous media, as it gives regular methods to deduce the equations governing the behavior of the medium from the propagation of singularities in it.

With this paper we would like to start a systematic investigation of the correspondence $\mathcal{Y} \leftrightarrow \mathcal{Y}_\Sigma$ for some fundamental equations of mathematical physics. Our aim here is to deduce and to discuss the equations $\mathcal{Y}_\Sigma$ for some well known equations, supposing $\Sigma$ to be the geometric folding-type singularities described in [6,7].

In Section 2 we recall the general feature of fold-type singularities, and the subsidiary equations associated to them. We recognize in some of them the analog of the classical characteristic equations, we call them $k$-characteristic equations.

In Section 3 we study the $k$-characteristic equations for some classical field theory equations: Klein–Gordon, Schrödinger and Dirac. Here we will see that, since the $k$-characteristic equations depend on the symbol only, the singularities will not be sensitive to the mass, in the first and third case, or the time derivative term and potential in the second case. We find, however, that for the Klein–Gordon equation, one and two characteristics describe, respectively, the propagation of massless point-like and "focusing" objects.

In Section 4 we reconsider the same equations as reduction of homogeneous equations in an extended space. Once this is done, all the terms in the equation contribute to the symbol, and the folds then yield the correct equations of motion of the corresponding propagating objects.

In Section 5 we find the remaining subsidiary equations (for 1-singularity, called complementary equations) for Klein–Gordon, Schrödinger, Dirac and Maxwell equations.

As for now, our paper is a sort of *'phenomenological'* paper, i.e. we discuss several aspects of singularities for relevant equations even though at the moment some equations do not allow for a clear cut physical interpretation.

## 2. Generalities

We recall that geometric singularities are singularities of multi-valued solutions of PDE's [2–5]. To make more precise these concepts some preliminaries are to be done. Let $E$ be a $(m+n)$-dimensional manifold (the manifold of all dependent and independent variables). Given two $n$-dimensional submanifolds, $L_1$ and $L_2$, of $E$ we say that they have the same *$k$-th order jet* at a point $\alpha \in L_1 \cap L_2$ iff they are tangent to each other with order $k$. So, a $k$-th order jet at $\alpha \in E$ is an equivalence class of $n$-dimensional submanifolds of $E$ passing through $\alpha$. The set of all such $k$-th order jets admits in a natural way a smooth manifold structure which is called the $k$-th order jet space of $n$-dimensional submanifolds of $E$ and is denoted by $J^k = J^k(E,n)$. Projections $J_{k,l} : J^k \to J^l$ are defined in a natural way. Let $(x,u)$ with $x = (x^1,\dots,x^n)$, $u = (u^1,\dots,x^m)$ be a divided local chart on $E$, that is a local chart on $E$ where some of the coordinate functions are proclaimed 'independent' variables and the remaining ones 'dependent' variables. Such a divided chart on $E$ generates a local chart on $J^k(E,n)$ composed of the variables

$$x^\mu, u^i, \dots, u^i_\sigma, \dots, \qquad |\sigma| \le k \tag{1}$$

with $1 \le \mu \le n$, $1 \le i \le m$ and $\sigma = (i_1,\dots,i_n)$ being a multiindex, $|\sigma| = i_1 + \cdots + i_n$. The *Cartan distribution* on $J^k(E,n)$, also called the *$k$-th order contact structure*, is defined as a distribution of tangent subspaces of $E$ given by the system of Pfaff equations

$$du^i_\sigma - u^i_{\sigma+1\,\mu} dx^\mu = 0, \tag{2}$$

with $1 \le i \le m$, $1 \le \mu \le n$, $|\sigma| < k$. Every $n$-dimensional submanifold $L$ of $E$ given in the form

$$u^i = f_i(x^1,\dots,x^n), \qquad 1 \le i \le m, \tag{3}$$

can be lifted canonically on $J^k(E,n)$. This lifted submanifold $L_{(k)} \subset J^k(E,n)$ is given by the equations

$$u^i_\sigma = \frac{\partial^{|\sigma|} f_i}{\partial x_\sigma}, \qquad 1 \le i \le m, \quad 0 \le |\sigma| \le k \tag{4}$$

where $\partial^{|\sigma|}/\partial x_\sigma$ stands for $\partial^{|\sigma|}/\partial x_1^{i_1} \cdots \partial x_n^{i_n}$ supposing that $\sigma = (i_1,\dots,i_n)$. A submanifold $N \subset J^k$ is called integral if it satisfies (2). Note that all the manifolds of the form $L_{(k)}$ are integral. An $n$-dimensional submanifold $N \subset J^k$

is called *R-manifold* if for almost every point $\theta \in N$ there exists a neighborhood of $\theta$ in $N$ which is of the form $L_{(k)}$ for an $L \subset E$. Here 'almost every' means excluding a subset $Y$ of $N$ with dim $Y < n$. It can be proved that this subset $Y$ coincides with the singular set of the projection $\pi_{k,k-1} : J^k \to J^{k-1}$ restricted to $N$. Because of this reason $Y$ is denoted by $\mathrm{sing}_N \subset N$. Let now $\theta$ be a point of $\mathrm{sing}_N$, $N$ being an $R$-manifold, and $T_\theta N$ be the tangent space of $N$ at $\theta$. The kernel of the projection of $T_\theta N$ along $\pi_{k,k-1}$ is called the *label* of $\theta$. These labels can be classified naturally with respect to the group of contact diffeomorphisms of $J^k$. Recall that contact diffeomorphisms are those diffeomorphisms that preserve the Cartan distribution of $J^k$. The result of this classification (see [6,7]) tells us that the label equivalence classes can be labeled by the finite-dimensional commutative $R$-algebras (in fact this result was formulated in [6] in slightly different terms).

As is well known a finite-dimensional commutative $R$- algebra splits in an essentially unique way into a direct sum of algebras $F_{(k)}$, with $F = R, C$, and $F_{(k)}$ stands for the $F$-algebra generated by an element, say $\xi$, subjected to the conditions $\xi^k = 0, \xi^{k-1} \neq 0$. In this paper we are concerned with solution singularities corresponding to $R_{(k)}$-label type which we will call folds. These singularities can be paralleled with the Thom–Boardman ones of the standard singularity theory, commonly denoted by $\Sigma_{(k)}$.

Recall, finally, that a $k$-th order system of PDE imposed on a $n$-dimensional submanifold of $E$ can be represented as a submanifold $\mathcal{Y} \subset J^k(E,n)$. In fact, local equations of $\mathcal{Y}$ are obviously of the form

$$F_j(\ldots x^\mu \ldots u^i \ldots u^i_\sigma \ldots) = 0, \qquad j = 1, \ldots, l \tag{5}$$

It is easily seen now that the functions $u^i = f_i(x)$, $i = 1, \ldots, m$ give us a solution of (5) iff $L_{(k)} \subset \mathcal{Y}$ where $L \subset E$ is the submanifold of $E$ given by the equations $u^i = f_i(x)$. This motivates the following concept which is crucial for what follows: an $R$-manifold $N$ is called a *multivalued solution* of (5) iff $N \subset \mathcal{Y}$.

We stress that the concept of $R$-manifold allows one to generalize the notion of solution for an arbitrary non-linear solution system of PDE essentially in the same way as the concept of lagrangian submanifold in $T^*M$ does for the Hamilton–Jacobi equation.

Roughly speaking, equations $\mathcal{Y}_\Sigma$, as mentioned in the introduction, describe possible shapes of singular submanifolds of $\mathrm{sing}_L$ formed by all $\Sigma$-type singular points. The system $\mathcal{Y}_\Sigma$ for $\Sigma = R_{(k)}$ will be called the *k-singularity system* associated to $\mathcal{Y}$. This is a (generally undetermined) system of partial differential equations on $n - k$ independent variables, which contains a specific equation which we call *k-characteristic*; 1-characteristic equations coincide with classical characteristic equations introduced by Hadamard when studying the uniqueness of the Cauchy problem. Note that the eikonal equation is the characteristic equation for a number of fundamental equations of mathematical physics. So,

$k$-characteristic equations for $k > 1$ describe, in particular, 'wave front' propagation for 'focusing objects'. We call *complementary equations* those which have to be added to the characteristic ones to get the full $k$-singularity system. Remembering that the characteristic equations describe the space–time form of solution singularities it is natural to think that complementary equations describe behaviors of internal structures of singularities giving a more intrinsic description. It is worth emphasizing that the study of asymptotic solutions of a differential equation leads to the theory of lagrangian submanifolds on $T^*M$. From his point of view one can treat lagrangian submanifolds as the asymptotic counterpart of $R$ manifolds. A physical interpretation of these complementary equations depends obviously on the physical nature of the original equation in question. We hope to present some examples of this kind in a future publication.

In the following two sections we deduce both $k$-characteristic equations and complementary equations for fundamental equations of mathematical physics. The necessary computational algorithms, extracted from the geometrical description of $\mathcal{Y}_\Sigma$ given in [7] are presented here without proof.

## 3. *k*-characteristic equations

### 3.1. Characteristic equations for a differential equation

The simplest, but non-trivial, case in which $k$-characteristic equations appear is that of second order scalar differential equations. $k$-characteristic equations for them can be found as follows.

Let $x = (x_1, \ldots, x_n)$ be independent variables. The general second order scalar differential equation is of the form:

$$F(x, u, u_i, u_{ij}) = 0 \tag{6}$$

where $u_i = \partial u / \partial x_i$ etc. The corresponding characteristic matrix is then of the form:

$$\mathcal{M} = \left( \frac{\partial F}{\partial u_{ij}} \right) . \tag{7}$$

With this matrix we can associate a bilinear pairing on 1-forms on the space of independent variables, namely

$$\langle du | dw \rangle_F = \frac{\partial F}{\partial u_{ij}} u_i w_j . \tag{8}$$

This pairing can be extended to the full exterior algebra. For instance on two-forms

$$\langle d\phi_1 \wedge d\phi_2 | d\phi_3 \wedge d\phi_4 \rangle_F = \langle d\phi_1 | d\phi_3 \rangle_F \langle d\phi_2 | d\phi_4 \rangle_F - \langle d\phi_1 | d\phi_4 \rangle_F \langle d\phi_2 | d\phi_3 \rangle_F . \tag{9}$$

The $k$-characteristic equations express the fact that the $(n-k)$-vector tangent to $N$ is isotropic with respect to the metric on $\Lambda^{n-k}TM$ induced naturally by the metric on $TM$ which is in its turn dual to the metric $M_{ij}$, $M_{ij} = \partial F/\partial u_{ij}$ on $T^*M$. Equivalently these equations state that the dual $k$-covector is isotropic with respect to the metric on $\Lambda^k T^*M$ coming from the just mentioned metric on $T^*M$. The fact that a decomposable $k$-covector $\theta_1^* \wedge \cdots \wedge \theta_k^* \in \Lambda^k T_a^* M$, where $\theta_1^*, \ldots, \theta_k^* \in T_a^* M$, is isotropic means that the $K$-dimensional subspace $L$ generated by the $\theta$'s is tangent to the characteristic cone $K_a^* \subset T_a^* M$ given by the equation

$$\sum_{ij} M_{ij}(x) p_i p_j = 0 \quad \text{for } x = a. \tag{10}$$

Similarly, the dual $(n-k)$-vector $\theta_1 \wedge \cdots \wedge \theta_{n-k} \in \Lambda^{n-k} T_a M$ , $\theta_1, \ldots, \theta_{n-k} \in T_a M$, being isotropic is tangent to the dual cone $K_a \subset T_a M$ given by

$$\sum M^{ij}(x) v_i v_j = 0 \tag{11}$$

where $M^{ij}$ is the $n-1$-order minor of the matrix $||M_{ij}||$ which is the complement of the element $M_{ij}$. It results that a solution of Eq. (17) is an $(n-k)$-dimensional submanifold $N$ of $M$ which is tangent to the cone $K_a$ at each point $a$. The lines along which $N$ is tangent to the cones $K_a$ form a field of directions (= one-dimensional distribution) on $N$. Integral curves of this distribution are exactly those along which $R_k$-singularities propagate. For $k = 1$ they are classical bicharacteristics of the original Eqs. (6).

To find explicitly the $k$-characteristic equations divide the variables $x$ into two parts, say $\tau = (\tau_1, \ldots, \tau_{n-k})$ and $y = (y_1, \ldots, y_k)$, where for instance

$$\begin{aligned} \tau_i &= x_{k+i} \ , 1 \le i \le n-k \\ y_i &= x_i \ , 1 \le i \le k \ . \end{aligned} \tag{12}$$

Suppose that the projection of the $k$-singularity (which lies in $J^2$) on the $x$ space is of the form

$$\tilde{\Phi}_i = y_i - \phi_i(\tau) = 0, \quad i = 1, \ldots, k \ .$$

The $k$-form

$$d\tilde{\Phi}_1 \wedge d\tilde{\Phi}_2 \wedge \cdots \wedge d\tilde{\Phi}_k \tag{13}$$

defines the $k$-characteristic equations by setting

$$\langle d\tilde{\Phi}_1 \wedge \cdots \wedge d\tilde{\Phi}_k | d\tilde{\Phi}_1 \wedge \cdots \wedge d\tilde{\Phi}_k \rangle = 0. \tag{14}$$

Using the coordinates introduced above these equations can be written in the following way: consider the $(n-k) \times n$-matrix

$$\Phi = \left\| \frac{\partial x_i(\tau)}{\partial \tau_j} \right\| \tag{15}$$

with $i = 1, \ldots n$, $j = 1 \ldots n-k$. Indicating with $\phi_{k,l} = \partial \phi_k / \partial \tau_l$, with the above choice (12) we have

$$\Phi = \begin{pmatrix} \phi_{1,1} & \phi_{2,1} & \ldots & \phi_{k,1} & 1 & 0 & \ldots & 0 \\ \phi_{1,2} & \phi_{2,2} & \ldots & \phi_{k,2} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi_{1,n-k} & \phi_{2,n-k} & \ldots & \phi_{k,n-k} & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{16}$$

Indicating by $\Phi_{i_1,\ldots,i_{n-k}}$, $1 \le i_1 < i_2 < \cdots < i_{n-k} \le n$, the minors of $\Phi$ composed of its $i_1$-th,...,$i_{n-k}$-th columns multiplied by $(-1)^{i_1+\cdots+i_k+n-k}$, then (14) takes the form

$$\Phi_{i_1,\ldots,i_{n-k}} \Phi_{j_1,\ldots,j_{n-k}} M^{i_1,\ldots,i_{n-k}}_{j_1,\ldots,j_{n-k}} = 0 , \tag{17}$$

where $M^{i_1,\ldots,i_{n-k}}_{j_1,\ldots,j_{n-k}}$ stands for the minor of $\mathcal{M}$ which remains after cancelling the $i_1, \ldots, i_{n-k}$-th rows and the $j_1, \ldots, j_{n-k}$-th columns; a sum over repeated indices is understood here and in the following . In some cases below we have found it more convenient to choose the $\tau$'s in a way different from (12), in these cases the explicit form of the $\Phi$ in (16) will change accordingly.

### *3.2. 1- and 2-characteristic equations for systems of differential equations.*

Let

$$F_i(x^\mu, u^a, u^a_\sigma) = 0, \quad i = 1, \ldots, l, \; a = 1, \ldots, m, \\ \mu = 1, \ldots, n, \; \sigma = (i_1, \ldots, i_n) \tag{18}$$

be a determined system $(l = m)$, and let $D(p)$, $p = (p_1, \ldots, p_n)$, be its characteristic determinant, that is:

$$D(p) = \det \left\| \frac{\partial F_i}{\partial u^a_\sigma} p^\sigma \right\|, \qquad p^\sigma = p_1^{i_1} \cdots p_n^{i_n}. \tag{19}$$

We now discuss one and two-folds.

#### *3.2.1. 1-Folds*

The singularity is of the form

$$f(x) = 0, \tag{20}$$

and the characteristic equation is

$$D(\nabla f) = 0. \tag{21}$$

*3.2.2. 2-Folds*

Let us look now to the $(n-2)$-dimensional surfaces corresponding to the 2-fold singularities in the form

$$x^1 = \phi(t)\ ,\ x^2 = \psi(t)\ ,\ t = (t^1, \dots, t^{n-2}). \tag{22}$$

Below we present only a procedure to deduce 2-characteristic equations, a more detailed presentation can be found in [7]. Let us do the substitutions

$$p_1 \to \xi,\ \ p_2 \to \eta,\ \ p_{2+\alpha} \to -(\xi\phi_\alpha + \eta\psi_\alpha) \tag{23}$$

where $\phi_\alpha = \partial\phi/\partial t_\alpha$, $\psi_\alpha = \partial\psi/\partial t_\alpha$. We obtain

$$D(p) = V(\xi, \eta). \tag{24}$$

Here $V(\xi,\eta)$ is a homogeneous polynomial of order $k$ (the order of the system $F_i = 0$) whose coefficients depend on $\phi_\alpha, \psi_\alpha$. The characteristic equation for 2-folds is then

$$r(h) = 0 \tag{25}$$

where $h = V(\xi, 1)$ and $r(h) = R(h, h')$. $R(h, g)$ being the resolvent determinant of the two-polynomial $f, g$:

$$h(\xi) = a_0\xi^n + a_1\xi^{n-1} + \cdots + a_n, \tag{26}$$

$$g(\xi) = b_0\xi^m + \cdots + b_m. \tag{27}$$

That is

$$R(h,g) = \begin{vmatrix} a_0 & a_1 & \dots & a_n & 0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & a_{n-1} & a_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & a_0 & a_1 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_0 & b_1 & \dots & b_m & 0 & 0 & \dots & 0 \\ 0 & b_0 & \dots & b_{m-1} & b_m & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 0 & b_0 & \dots & b_m \end{vmatrix}. \tag{28}$$

The determinant is of order $n + m$. If the system $F_i = 0$ is overdetermined $(l > m)$, the 1- and 2-characteristic equations are (21) and (25) imposed on all characteristic determinants.

## 4. Examples

We now discuss in detail some examples, first the case of the Klein–Gordon equation. Here we find the eikonal equation for 1-folds. For 2-folds we find an equation describing a null two dimensional surface, to be interpreted as an analog of 'wave front' propagation. The situation is analogous for 3-folds, where we will observe the $k, (n-k)$-fold duality. Then we discuss the Schrödinger equation where we find that, since the symbol of the differential operator does not contain any information, not only on the potential $V$, but also on the time derivatives, the solutions of the characteristic equation are 'space-like', that is, transverse with respect to time. We will discuss and interpret these results. Finally we will consider 1- and 2-characteristic equations for the Dirac equation.

*4.1. Klein–Gordon equation*

The Klein–Gordon equation is:

$$(\partial_t^2 - \nabla^2 + m^2)u = 0 \tag{29}$$

The matrix $\mathcal{M}$ and the differential equation on $J_2$ are respectively

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{30}$$

and

$$F = u_{00} - \Sigma u_{ii} + m^2 u = 0. \tag{31}$$

The parametrizations of the singularities and the corresponding characteristic equations for 1,2 and 3-folds are as follows.

*4.1.1. 1-Folds*

We write the singularity in the form:

$$t = \phi(x_1, x_2, x_3) \ , \tag{32}$$

therefore the $3 \times 4$ matrix $\Phi$ is

$$\Phi = \begin{pmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & 1 & 0 \\ \phi_3 & 0 & 0 & 1 \end{pmatrix} . \tag{33}$$

Using Eq. (17), or Eq. (14)

$$\langle d\tilde{\Phi} | d\tilde{\Phi} \rangle = 0 \tag{34}$$

where

$$\tilde{\Phi} = \phi(x_1, x_2, x_3) - t = 0, \tag{35}$$

we obtain

$$\sum_{i=1}^{3} \phi_i^2 = (\nabla\phi)^2 = 1 . \tag{36}$$

This is the eikonal equation. Observe however that an interpretation of this equation (more precisely of its characteristics) in terms of particles associated to the fields is correct only for $m = 0$ as the surfaces move at the speed of light. The meaning of (36) for the massive case has been discussed by Racah [9] in the context of the Dirac equation. He observed that if the wave function of a particle is different from zero in a finite region, then the eikonal equation describes the motion of the boundary of such a region. Since the Fourier expansion of the wave function will have components with all possible wave numbers, this surface will move at the speed of light, even if the particle does not. This problem of interpretation will obviously be present for all $k$-folds, as well as for the Schrödinger case below; the next section is in fact dedicated to a discussion of this problem.

#### *4.1.2. 2-Folds*

In this case it is convenient to parametrize the singularity as follows:

$$x_2 = \phi_2(t, x_1) \tag{37}$$

$$x_3 = \phi_3(t, x_1) . \tag{38}$$

As this is not the choice made in (12), the form of $\Phi$ will be slightly different; using Eq. (15) we obtain:

$$\Phi = \begin{pmatrix} 1 & 0 & \phi_{2,0} & \phi_{3,0} \\ 0 & 1 & \phi_{2,1} & \phi_{3,1} \end{pmatrix} . \tag{39}$$

And the equation is

$$(\phi_{2,0}\phi_{3,1} - \phi_{2,1}\phi_{3,0})^2 + (\phi_{3,0})^2 + (\phi_{2,0})^2 - (\phi_{3,1})^2 - (\phi_{2,1})^2 - 1 = 0 \tag{40}$$

This equation describes a two dimensional null submanifold, that is, a surface which is everywhere tangent to a null cone. Notice that the world surfaces of null (tensionless) strings [8] are two dimensional null submanifolds.

#### *4.1.3. 3-Folds*

Again here it is convenient to use $t$ in the parametrization of the singularity:

$$x_1 = \phi_1(t) \tag{41}$$

$$x_2 = \phi_2(t) \tag{42}$$

$$x_3 = \phi_3(t) , \tag{43}$$

thus

$$\Phi = (1 \quad \phi_{1,0} \quad \phi_{2,0} \quad \phi_{3,0}) \,. \tag{44}$$

And the characteristic equation is

$$\sum_{i=1}^{3} \phi_{i,0}^2 = 1 \tag{45}$$

This equation describes a null curve. In this equation we immediately recognize the lagrangian of a free particle, thus showing the above mentioned duality.

### 4.2. Schrödinger equation

The Schrödinger equation is:

$$\left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2\nabla^2}{2m} + V(\boldsymbol{x})\right) u(\boldsymbol{x},t) = 0 \tag{46}$$

with $t = x_0$. Written in terms of the coordinates on $J_2$ this equation becomes

$$F = -i\hbar u_0 - \frac{\hbar^2}{2m}\sum_{i=1}^{3} u_{ii} + Vu = 0. \tag{47}$$

From Eq. (7) we obtain for the matrix $\mathcal{M}$:

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hbar^2/2m & 0 & 0 \\ 0 & 0 & -\hbar^2/2m & 0 \\ 0 & 0 & 0 & -\hbar^2/2m \end{pmatrix} \tag{48}$$

#### 4.2.1. 1-Folds

The singularity is of the form:

$$t = \phi(x_1, x_2, x_3)\,; \tag{49}$$

therefore the $3 \times 4$ matrix $\Phi$ is

$$\Phi = \begin{pmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & 1 & 0 \\ \phi_3 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

and using Eq. (17) we obtain

$$\sum_{i=1}^{3} \phi_i^2 = 0 \tag{51}$$

which has as solution

$$\phi = t = \text{const.}, \tag{52}$$

that is, the fold is transverse with respect to time. This is in agreement with what we said about the eikonal equation for the massive Klein–Gordon equation. Here in fact, the theory being non-relativistic, the surface bounding the region in which the wave function is different from zero moves with infinite speed.

#### 4.2.2. 2-Folds

For 2-folds instead the equation for the form of the singularity and of the matrix $\Phi$ are:

$$t = \phi_0(x_2, x_3) \tag{53}$$

$$x_1 = \phi_1(x_2, x_3) \tag{54}$$

$$\Phi = \begin{pmatrix} \phi_{0,2} & \phi_{1,2} & 1 & 0 \\ \phi_{0,3} & \phi_{1,3} & 0 & 1 \end{pmatrix} \tag{55}$$

and the equation is

$$(\phi_{0,2}\phi_{1,3} - \phi_{1,2}\phi_{0,3})^2 + \phi_{0,2}^2 + \phi_{0,3}^2 = 0\ . \tag{56}$$

The solutions are:

$$\begin{aligned} t &= \text{constant} \\ x_1 &= \phi_1(x_2, x_3) \end{aligned} \tag{57}$$

where $\phi_1(x_2, x_3)$ is an arbitrary function. They describe a two dimensional surface in space at a fixed time. Even in this case the singularities are transverse with respect to time.

#### 4.2.3. 3-Folds

We parametrize the 3-folds as follows:

$$t = \phi_0(x_3) \tag{58}$$

$$x_1 = \phi_1(x_3) \tag{59}$$

$$x_2 = \phi_2(x_3), \tag{60}$$

thus

$$\Phi = (\phi_{0,3} \quad \phi_{1,3} \quad \phi_{2,3} \quad 1) \tag{61}$$

and the equation is

$$(\phi_{0,3})^2 = 0 \tag{62}$$

which again has solution $\phi_0 = t$ =const., the other $\phi$'s being arbitrary. This describes a one-dimensional curve in space at a given time. And therefore such a curve cannot be considered a world-line.

### 4.3. Dirac equation

We finish this section with a brief discussion of the Dirac equation, or rather the Dirac equations, as it is a system of four equations, one for each component of the spinor. The system is

$$(i\partial\!\!\!/ - m)u = 0 \tag{63}$$

which written explicitly is:

$$F_\alpha(x^\mu, u^\alpha, u^\alpha_\mu) = i\gamma^\mu_{\alpha\beta} u^\alpha_\mu - m\delta_{\alpha\beta} u^\beta = 0. \tag{64}$$

#### 4.3.1. 1-Folds

If the singularity is defined by

$$f(x^\mu) = 0, \tag{65}$$

the equation characteristic is:

$$\det\left(\frac{\partial F_\alpha}{\partial u^\beta_\nu} f_\nu\right) = 0, \tag{66}$$

which gives:

$$f_\nu f^\nu = 0. \tag{67}$$

#### 4.3.2. 2-Folds

Parametrizing the fold by

$$x_2 = \phi_2(x_0, x_1) \tag{68}$$

$$x_3 = \phi_3(x_0, x_1) \tag{69}$$

and following the procedure of Section 3 we find the 2-characteristic equation:

$$-\sum_i \phi_{2,i}\phi_{3,i} + \sum_{i,j} \phi^2_{2,i}\phi^2_{3,i}\phi^2_{2,j} + \sum_{i\neq j} \phi^2_{2,i}\phi^2_{3,j} = 0. \tag{70}$$

## 5. Extended equations

In the case of the Maxwell equations, or massless Klein–Gordon, the characteristic equation describes the classical motion of the particles associated to the fields (photons or scalar massless particles). In the Schrödinger or massive

Klein–Gordon case this does not happen. The reason is that the characteristic equation is sensitive only to the symbol and therefore potential and time derivative do not appear in the former case, while all information about the mass is absent from the latter, for which wave fronts move in fact at the speed of light.

We expect that other singularities, sensitive not only to the symbol, will provide the particle trajectories even in this case. We observe however that it is possible to write the equations above as reduction of homogeneous equations in an extended space. If this is done, all the terms in the equation will contribute to the symbol, and the folds will then yield equations of motion of the particles even in these cases. We will describe briefly the reduction procedure and then consider again the fold singularities and how they get reduced.

Consider a second order differential equation of the kind

$$(A^{\mu\nu}\partial_\mu\partial_\nu + B^\mu\partial_\mu + C)u = 0 \tag{71}$$

where the coefficients $A^{\mu\nu}, B^\mu, C$ are functions, and $\mu, \nu = 0, \ldots, n-1$. Introducing an extra variable $x_{-1}$, this equation can be obtained as reduction of the following equation homogeneous in the second derivatives:

$$g^{ab}\partial_a\partial_b\tilde{u} = 0, \quad a, b = -1, \ldots, 3 \tag{72}$$

The $n+1 \times n+1$ metric $g^{ab}$, written in terms of the matrix $A = (A_{\mu\nu})$, the vector $B = (B_\mu)$ and the scalar $C$, has the form:

$$g^{ab} = \begin{pmatrix} C & B \\ B^T & A \end{pmatrix} \tag{73}$$

If we consider the space with an additional variable as a principal $R$-bundle, functions $\tilde{u}$ on the total space are simply $R$-equivariant functions. The new operator $\tilde{D}$ on the total space and the operator $D$ on the base manifold are related by

$$g^*(\pi^*(Du)) + \tilde{D}(g^*\pi^*u). \tag{74}$$

The reduction is obtained restricting the dependence of the functions $\tilde{u}(x_a)$ on the $x_{-1}$ as follows:

$$\tilde{u}(x_a) = e^{x_{-1}}u(x_\mu) \tag{75}$$

and setting $x_{-1}$ equal to a constant. In our examples, in order to have that the extended metric has signature $(+, -, \ldots, -)$, we will use Eq. (73) with $C$ replaced by $-C$ and Eq. (75) with $e^{x_{-1}}$ replaced by $e^{ix_{-1}}$.

The reduction for the singularity is obtained by fixing the value of the variable $x_{-1}$. The introduction of this reduction procedure [10] here may seem artificial, it is nonetheless interesting because it shows that already the simplest (fold) singularities have several non-trivial features and are able to capture relevant aspects of the dynamics of particles associated to fields. In the following we will find it convenient to put some constants in the exponent of $\tilde{u}$ in (75) rather than in the metric $g^{ab}$.

### 5.1. Extended Klein–Gordon case

The extended Klein–Gordon equation is:

$$g^{ab}\tilde{u}_{ab} = 0 \tag{76}$$

with the metric

$$g^{ab} = \begin{pmatrix} -m^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{77}$$

and the reduction follows from:

$$\tilde{u} = e^{ix_{-1}}u \tag{78}$$

#### 5.1.1. 1-Folds

Parametrize the singularity by:

$$x_{-1} = \phi(x_\mu), \qquad \mu = 0, \ldots, 3 \tag{79}$$

and $\Phi$ is

$$\Phi = \begin{pmatrix} \phi_0 & 1 & 0 & 0 & 0 \\ \phi_1 & 0 & 1 & 0 & 0 \\ \phi_2 & 0 & 0 & 1 & 0 \\ \phi_3 & 0 & 0 & 0 & 1 \end{pmatrix}; \tag{80}$$

the characteristic equation then is:

$$g^{\mu\nu}\phi_\mu\phi_\nu - m^2 = 0 \tag{81}$$

which after reduction, namely setting $x_{-1}$ = const. in (79), now correctly describes free particles with arbitrary masses.

#### 5.1.2. 2-Folds

The singularity and $\Phi$ are:

$$x_{-1} = \phi_{-1}(x_i) \tag{82}$$

$$x_0 = \phi_0(x_i), \qquad i = 1, 2, 3 \tag{83}$$

$$\Phi = \begin{pmatrix} \phi_{-1,1} & \phi_{0,1} & 1 & 0 & 0 \\ \phi_{-1,2} & \phi_{0,2} & 0 & 1 & 0 \\ \phi_{-1,3} & \phi_{0,3} & 0 & 0 & 1 \end{pmatrix} \tag{84}$$

and the characteristic equation is:

$$\sum_i (\sum_{j,k} \varepsilon_{ijk}\phi_{-1,j}\phi_{0,k})^2 + \sum_i \left( m^2(\phi_{0,i})^2 - (\phi_{-1,i})^2 \right) - m^2 = 0 \tag{85}$$

*5.1.3. 3-Folds*

The singularity is parametrized by

$$x_{-1} = \phi_{-1}(x_i) \tag{86}$$

$$x_0 = \phi_0(x_i) \tag{87}$$

$$x_1 = \phi_1(x_i), \qquad i = 2, 3, \tag{88}$$

$$\Phi = \begin{pmatrix} \phi_{-1,2} & \phi_{0,2} & \phi_{1,2} & 1 & 0 \\ \phi_{-1,3} & \phi_{0,3} & \phi_{1,3} & 0 & 1 \end{pmatrix} \tag{89}$$

The characteristic equation is:

$$\begin{aligned} &(\phi_{-1,2}\phi_{0,3} - \phi_{0,2}\phi_{-1,3})^2 + (\phi_{-1,2}\phi_{1,3} - \phi_{1,2}\phi_{-1,3})^2 \\ &\quad - m^2(\phi_{0,2}\phi_{1,3} - \phi_{1,2}\phi_{0,3})^2 + (\phi_{-1,3})^2 + (\phi_{-1,2})^2 \\ &\quad + m^2\left( (\phi_{1,3})^2 + (\phi_{1,2})^2 - (\phi_{0,3})^2 - (\phi_{0,2})^2 + 1 \right) = 0. \end{aligned} \tag{90}$$

*5.1.4. 4-Folds*

In this case we parametrize the singularity with $t$:

$$x_{-1} = \phi_{-1}(t) \tag{91}$$

$$x_i = \phi_i(t), \qquad i = 1, 2, 3 \tag{92}$$

$$\Phi = (\phi_{-1,0} \quad 1 \quad \phi_{1,0} \quad \phi_{2,0} \quad \phi_{3,0}) \tag{93}$$

and the characteristic equation is:

$$(\phi_{-1,0})^2 = m^2 \left( 1 - \sum_{i=1}^{3} (\phi_{i,0})^2 \right), \tag{94}$$

which after reduction gives the square of the usual lagrangian $m\sqrt{x_\mu x^\mu}$.

We now list the parametrization of the singularities and the corresponding characteristic equations for $k$-folds in the Schrödinger case.

## 5.2. Extended Schrödinger case

The extended Schrödinger equation is:

$$g^{ab}\tilde{u}_{ab} = 0, \qquad a, b = -1, 0, 1, 2, 3 \tag{95}$$

with

$$\|g^{ab}\| = \begin{pmatrix} -V & -1/2 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2m & 0 & 0 \\ 0 & 0 & 0 & -1/2m & 0 \\ 0 & 0 & 0 & 0 & -1/2m \end{pmatrix} \tag{96}$$

The expression for $\tilde{u}$ is

$$\tilde{u}(x) = e^{ix_{-1}/\hbar} u(x^{\mu}). \tag{97}$$

### 5.2.1. 1-Folds

$$x_{-1} = \phi(x_{\mu}) \tag{98}$$

$$\frac{-1}{2m}(\nabla\phi)^2 + \phi_0 - V = 0 \tag{99}$$

Reducing $\phi$ one finds that this can be interpreted in four dimensions as the Hamilton–Jacobi equation.

### 5.2.2. 2-Folds

$$x_{-1} = \phi_{-1}(x_i) \tag{100}$$

$$x_0 = \phi_0(x_i), \quad i = 1,2,3 \tag{101}$$

$$\left(\frac{1}{2m}\right)^2 \sum_{ijk} (\epsilon_{ijk}\phi_{-1,j}\phi_{0,k})^2 - \frac{1}{2m}\sum_i \phi_{-1,i}\phi_{0,i} + V\frac{1}{2m}\sum_i (\phi_{0,i})^2 - \frac{1}{4} = 0 \tag{102}$$

### 5.2.3. 3-Folds

$$x_{-1} = \phi_{-1}(x_2, x_3) \tag{103}$$

$$x_0 = \phi_0(x_2, x_3) \tag{104}$$

$$x_1 = \phi_1(x_2, x_3) \tag{105}$$

$$-\left(\frac{1}{2m}\right)^3 \left(\phi_{-1,2}\phi_{0,3} - \phi_{0,2}\phi_{-1,3}\right)^2 + \left(\frac{1}{2m}\right)^2 \Big(\left(\phi_{0,2}\phi_{1,3} - \phi_{1,2}\phi_{0,3}\right)\left(\phi_{-1,2}\phi_{1,3} - \phi_{1,2}\phi_{-1,3}\right) + \phi_{-1,3}\phi_{0,3} + \phi_{-1,2}\phi_{0,2}\Big) - V\left(\frac{1}{2m}\right)^2 \Big(\left(\phi_{0,2}\phi_{1,3} - \phi_{1,2}\phi_{0,3}\right)^2 + \left(\phi_{0,3}\right)^2 + \left(\phi_{0,2}\right)^2\Big)$$

$$-\frac{1}{8m}\left((\phi_{1,3})^2 + (\phi_{1,2})^2 + 1\right) = 0 \tag{106}$$

*5.2.4. 4-Folds*

$$x_{-1} = \phi_{-1}(t) \tag{107}$$

$$x_i = \phi_i(t), \quad i = 1\dots 3 \tag{108}$$

$$\tfrac{1}{2}m\left(\sum_{i=1}^{3}(\phi_{i,0})^2\right) - \phi_{-1,0} + V = 0 \tag{109}$$

After reduction this equation gives the lagrangian of a particle in the potential $V$.

## 6. Complementary equations for 1-folds

In this last section we discuss the complementary equations which, as we said, are necessary for a complete description of the singularities.

The procedure for finding the complementary equations for 1-folds in the case of 1-singularity equations for a scalar second order differential equation is as follows. Let us consider a singularity described by the equation

$$x_n - \phi(x_i) = 0, \quad i = 1,\dots,n-1\ , \tag{110}$$

a basis of vector fields tangent to the projection of the fold on the base is

$$X_i = \partial_i + \phi_i\partial_n \tag{111}$$

We also have the initial data

$$u - h = 0 \tag{112}$$

$$u_n - g = 0 \tag{113}$$

Initial data on a singularity are subject to constraints, which can be described in terms of a set of equations, which turn out to be the complementary equations. To find them let us proceed as follows.

Acting with the vector fields $X_i$ on the initial data, after some manipulations and including the differential equations $F = 0$, we obtain a system in the unknowns $u_{\mu\nu}$:

$$\begin{aligned} u_{in} + \phi_i u_{nn} &= g_i \\ u_{ij} - \phi_i\phi_j u_{nn} &= h_{ij} - \phi_{ij}g - (\phi_i g_j + \phi_j g_i) \\ F(x_\mu, u, u_\mu, u_{\mu\nu}) &= 0 \end{aligned} \tag{114}$$

There are $n - 1 + \frac{1}{2}n(n-1) + 1 = \frac{1}{2}n(n+1)$ equations in the same number of unknowns.

Using the 1-characteristic equation (i.e. along the singularity) the system becomes degenerate, that is, the determinant of the matrix of the coefficients of the $u_{\mu\nu}$'s vanishes identically when the characteristic equation is substituted into it. Writing the system in matricial form we can express it as

$$M \cdot U = C \tag{115}$$

where $U$ is the vector of the unknowns, $M$ is the matrix of the coefficients, and $C$ is the vector of the known factors of the system (114). If we indicate by $Y_i$ a basis of vectors of the left kernel of $M$,

$$Y_i^\dagger M = 0, \tag{116}$$

the complementary equations are

$$Y_i^\dagger C = 0 \tag{117}$$

Obviously if we change the choice of the coordinate $x_n$ in (110), Eqs. (111)–(114) will change accordingly.

### 6.1. Examples

#### 6.1.1. Klein–Gordon case

The singularities are described by the equation

$$t - \phi(x_i) = 0 \ , \qquad i = 1, \ldots, 3 \ , \tag{118}$$

and the initial data are:

$$u - h = 0 \tag{119}$$

$$u_0 - g = 0 \tag{120}$$

The system (114) becomes:

$$\begin{aligned} u_{i0} + u_{00}\phi_i &= g_i \\ u_{ij} - \phi_i\phi_j u_{00} &= h_{ij} - \phi_{ij} g - (\phi_i g_j + g_j \phi_i) \\ u_{00} - \sum_{i=1}^{3} u_{ii} &= -m^2 h \end{aligned} \tag{121}$$

The characteristic equation is

$$(\nabla\phi)^2 = 1 \ . \tag{122}$$

The complementary equation is

$$\nabla^2 h + m^2 h - g - (\nabla^2\phi) g - 2\nabla\phi \cdot \nabla g = 0 \tag{123}$$

#### 6.1.2. Schrödinger case

With the same parametrization of the singularity, the system (114) becomes:

$$u_{i0} + u_{00}\phi_i = g_i$$
$$u_{ij} - \phi_i\phi_j u_{00} = h_{ij} - \phi_{ij}g - (\phi_i g_j + g_j\phi_i)$$
$$-\frac{\hbar^2}{2m}\sum_{i=1}^{3} u_{ii} = -Vu + i\hbar g \tag{124}$$

The characteristic equation is

$$\sum_{i=1}^{3}\phi_i^2 = 0\ , \tag{125}$$

whose solution is

$$\phi_i = 0\ . \tag{126}$$

The complementary equation is:

$$-\frac{\hbar^2}{2m}\nabla^2 h + Vu - i\hbar g = 0 \tag{127}$$

#### 6.1.3. Extended Klein–Gordon

Parametrizing the singularity as follows:

$$x_{-1} - \phi(x_\mu) = 0, \qquad \mu = 0,\dots,3\ , \tag{128}$$

the complementary equation is:

$$\Box\tilde{h} - \Box\phi\tilde{g} - 2\partial_\mu\phi\partial^\mu\tilde{g} = 0 \tag{129}$$

The reduced equation is obtained using for $\tilde{h}$ and $\tilde{g}$ the ansatz in Eq. (78):

$$\tilde{h}(x) = e^{ix_{-1}}h(x^\mu) \tag{130}$$
$$\tilde{g}(x) = \partial_{-1}\tilde{h}(x) = ieix_{-1}h(x^\mu). \tag{131}$$

The result is

$$\Box h - \Box\phi h - 2i\partial_\mu\phi\partial^\mu h = 0. \tag{132}$$

#### 6.1.4. Extended Schrödinger

Using the same parametrization of the previous example, the complementary equation is:

$$\frac{\hbar^2}{2m}(\nabla^2\tilde{h} - (\nabla^2\phi)\tilde{g} - 2\nabla\phi\cdot\nabla\tilde{g}) - \tilde{g}_0 = 0. \tag{133}$$

According to Eq. (97), we set

$$\tilde{h}(x) = e^{ix_{-1}/\hbar} h(x^{\mu}) \tag{134}$$

$$\tilde{g}(x) = \partial_{-1}\tilde{h}(x) = \frac{i}{\hbar} e^{ix_{-1}/\hbar} h(x^{\mu}), \tag{135}$$

so that the reduced equation is

$$\frac{\hbar^2}{2m}\left(\nabla^2 h - \frac{i}{\hbar}(\nabla^2\phi)h - \frac{i}{\hbar}2\nabla\phi\cdot\nabla h\right) - \frac{i}{\hbar}h_0 = 0 \tag{136}$$

*6.1.5. Maxwell case*

Now we deduce the complementary equations for Maxwell equations. This time the system of differential equations is degenerate of the first order, and the fields are vectors, therefore the fields $u^{\alpha}$ will have an extra (upper) index; to avoid confusion, we will use for it the first letters of the greek alphabet. The differential equations are represented on $J_1$ by $F_a(x_{\mu}, u^{\alpha}, u^{\alpha}_{\mu}) = 0$.

The characteristic equations can be obtained by equating to 0 highest order minors of the characteristic matrix: $(\partial F_a/\partial u^{\alpha}_{\mu}) f_{\mu}$ where the $f_{\mu}(x) = 0$ is the equation for the singularity.

In the Maxwell case the fields are 6, we identify $u^i = E^i, u^{i+3} = B^i$, $i = 1, 2, 3$; the 8 Maxwell equations are:

$$\begin{aligned} u^i_i &= 0 \\ u^{i+3}_i &= 0 \\ \varepsilon^{ijk}u^j_i &= -u^{k+3}_0 \\ \varepsilon^{ijk}u^{j+3}_i &= \epsilon\mu u^k_0 \end{aligned} \tag{137}$$

The characteristic matrix is rectangular $6 \times 8$, and its minors of order 6 have determinants of the form

$$\epsilon\mu f_0^2 - \sum_{i=0}^{3} f_i^2 = 0. \tag{138}$$

So the only solution is, as expected, the eikonal equation.

We now look for the complementary equations. To this purpose let us consider a singularity described by the equations

$$t - \phi(x^i) = 0, \tag{139}$$

with tangent fields spanned by

$$X_i = \partial_i + \phi_i\partial_0\ . \tag{140}$$

This time the initial data will be of the kind:

$$u^{\alpha} - h^{\alpha} = 0 \tag{141}$$

Proceeding in analogy with the scalar case we obtain:

$$u_i^\alpha + \phi_i u_0^\alpha = h_i^\alpha \tag{142}$$

which together with (137) form a system on $J_1$.

To find the complementary equations we can proceed as before, even if the system is overdetermined. In this case the matrix of the coefficients is $26 \times 24$ and, using the eikonal, the left kernel has dimension four. The equations one obtains can be written as:

$$\epsilon\mu\nabla \cdot h_E + \nabla f \cdot (\nabla \wedge h_B) = 0 \tag{143}$$

$$\epsilon\mu\nabla \wedge h_E + \nabla f (\nabla \cdot h_B) - \nabla f \wedge (\nabla \wedge h_B) = 0 \tag{144}$$

where

$$h_E = (h_1, h_2, h_3), \qquad h_B = (h_4, h_5, h_6) .$$

*6.1.6. Dirac equation*

Here we find four complementary equations:

$$f_\mu \gamma^\mu_{\nu\alpha} (i\gamma^j_{\alpha\beta} u^\beta_j - u^\alpha) = 0 \tag{145}$$

where $j = 1, \ldots, 3$, $\alpha, \beta, \mu, \nu = 1, \ldots, 4$, and $f = 0$ is the equation of the singularity.

## 7. Conclusions

From the mathematical point of view the theory of singularities of the generalized solutions of PDE's is a generalization of the standard singularity (or 'catastrophe') theory. In fact the latter can be viewed as the part of the former dealing with the solution of zero-order differential equations. Many interesting aspects appear in this generalization, and we discussed but a few of them in this paper. Therefore there is no doubt that this generalized theory of singularities is worth to be developed as a new branch of pure mathematics to a much larger extent. For the state of the art see refs. [3–6,11]. A possible important role of this theory is discussed in refs. [6,12]

The 'phenomenology' presented in this article indicates a number of more concrete problems of interest. Among them there is the systematic development of the theory of bicharacteristics of $k$-characteristic equations. Some sort of duality between $k$-characteristic and $(n-k)$-characteristic equations emerges from this paper; this led to the hypothesis of an analog of the Legendre transform, and a natural extension of the classical Lagrangian formalism. Apart from these arguments we can expect a generalization of the standard Hamiltonian formalism which, with respect to $\Sigma$-characteristic equations, would play the same role the standard one plays with the usual characteristics. It is very likely that such a generalization is in the spirit of refs. [13,14].

A problem that remains open is that of a systematical physical interpretation of the new equations presented here, as well as the search of alternative singularity types. In particular the extended equations presented in Section 5, which at this moment can seem a mere trick, have to be understood more conceptually. Another open question of possible physical relevance is that of putting the old problem of the field sources [15,16] in the framework of the theory presented here.

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